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Journal of Sound and Vibration 287 (2005) 203–226

JOURNAL OF
SOUND AND
VIBRATION

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A multi-step transversal linearization (MTL) method in non-linear structural dynamics

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Received 19 January 2004; received in revised form 22 September 2004; accepted 1 November 2004

Available online 2 February 2005

Abstract

An implicit family of multi-step transversal linearization (MTL) methods is proposed for efficient and numerically stable integration of nonlinear oscillators of interest in structural dynamics. The presently developed method is a multi-step extension and further generalization of the locally transversal linearization (LTL) method proposed earlier by Roy (Proceedings of the Academy of the Royal Society (London) 457 (2001) 539–566), Roy and Ramachandra (Journal of Sound and Vibration 41 (2001a) 653–679, International Journal for Numerical Methods in Engineering 51 (2001b) 203–224) and Roy (International Journal of Numerical Methods in Engineering 61 (2004) 764). The MTL-based linearization is achieved through a non-unique replacement of the nonlinear part of the vector field by a conditionally linear interpolating expansion of known accuracy, whose coefficients contain the discretized state variables defined at a set of grid points. In the process, the nonlinear part of the vector field becomes a conditionally determinable equivalent forcing function. The MTL-based linearized differential equations thus become explicitly integrable. Based on the linearized solution, a set of algebraic, constraint equations are so formed that transversal intersections of the linearized and nonlinearized solution manifolds occur at the multiple grid points. The discretized state vectors are thus found as the zeros of the constraint equations. Simple error estimates for the displacement and velocity vectors are provided and, in particular, it is shown that the formal accuracy of the MTL methods as a function of the time step-size depends only on the error of replacement of the nonlinear part of the vector field. Presently, only two different polynomial-based interpolation schemes are employed for transversal linearization, viz. the Taylor-like interpolation and the Lagrangian interpolation. While the Taylor-like interpolation leads to numerical ill-conditioning as the

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order of interpolation increases, the Lagrangian interpolation is shown to overcome this numerical problem. Finally, the family of MTL methods is illustrated through limited numerical results for a couple of harmonically driven workhorse oscillators, viz. the hardening Duffing and the Duffing–Holmes' oscillators. Comparisons with results obtained via a sixth-order Runge–Kutta method with adaptive time step-sizes are also provided.

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1. Introduction

Many of the available numerical integration techniques, applicable to nonlinear ordinary differential equations (ODEs) of relevance to structural dynamics, are constructed through explicit or implicit, Taylor-like expansions of the approximate solution vector in terms of a chosen step size, h (fixed or variable). Some of these techniques are Runge–Kutta method and its variations [25], the Newmark family of methods [1,2], the Adams–Bashforth and Adams–Moulton methods, finite difference schemes, geometric integrators (i.e., those preserving the given first integrals of motion) based on discrete gradients of functions [3] etc. A comprehensive review of some of these techniques and many others may be found in Richtmyer and Morton [4] and McLachlan et al. [5]. A specific difficulty of methods based on Taylor-series expansions is that information regarding the first or higher derivatives of the vector field is needed. In the context of finite element applications, one is required to repeatedly derive the tangent stiffness matrix (which is essentially the Jacobian of a part of the nonlinear vector field constructed around a known or guessed point in the space of the discretized state vectors) over each time step [6]. In addition to the extremely laborious nature of such an exercise, there are obvious problems in applying such approaches for problems with C^0 continuity. Another approach of recent interest is via Lie group theories [7,8]. Such approaches have been successfully applied to linear ODEs with explicitly time-dependent vector fields (such as parametrically excited linear problems) and they often make use of the Magnus or Fer expansions, which involve repeated Lie bracket operations and are quite cumbersome to compute except for the first two or three terms. Zanna [9] has even proposed a collocation method for nonlinear ODEs based on such expansions in another related development. Marthinsen [10] has considered polynomial interpolations in Lie groups and applied the concepts for integrating ODEs in terms of Lie algebra actions. An alternative, non-gradient, implicit approach, referred to as 'locally transversal linearization' (LTL), has recently been proposed by Roy [26], Roy and Ramachandra [11,12] and Ramachandra and Roy [13,14]. It completely avoids the usage of Taylor-like expressions. The LTL approach is a single-step procedure and reduces the nonlinear problem to an easily integrable linearized problem so that the vector fields for both the linearized and nonlinear problems remain identical at the forward discretization point, where an approximate solution vector needs to be found. Finally, a solution for the nonlinear problem is derived by finding the roots of a set of constraint algebraic equations, which in turn determine the point of transversal intersection (in the discretized state space) of the linearized and nonlinear solution manifolds.

The present study may be viewed as an effort to further generalize and improve upon the LTL procedure and thus proposed a multi-step extension of the procedure, herein referred to as the 'multi-step transversal linearization' (MTL) method. Unlike the LTL procedure, the nonlinear

part of the vector field is non-uniquely replaced through a set of interpolating basis functions, whose coefficients are expressible in terms of the discretized state variables over a multiple set of grid points. The nonlinear part of the vector field is thus effectively transformed into an equivalent and conditionally known forcing function. The choice of the interpolating basis functions is quite arbitrary and may be tailored to suit the physics of the underlying dynamical system or the type of solutions that the analyst needs to capture. In order to bring out the importance of the choice of such interpolating functions on the accuracy and successful implementation of the MTL method, two different polynomial-based interpolation schemes are presently adopted. The first scheme is an interpolating expansion similar to the Taylor series and the second scheme is directly based on a Lagrangian interpolation. The discretized state variables are then determined as the roots of a system of constraint algebraic equations, which ensure transversal intersections of linearized and nonlinear solution manifolds at the chosen set of multiple grid points. Since an exact solution of the transversally linearized problem is known (i.e., can be computed to virtually any desired degree of accuracy), it is shown that the formal accuracy of the MTL-based method as a function of the time step-size is only dependent upon the error of replacing the nonlinear part of the vector field through the interpolating basis set. Presently, a limited numerical illustration of the method is provided for a few low-dimensional nonlinear oscillators in their periodic and chaotic regimes. Comparisons with acceptable alternative solutions (obtained through a sixth-order Runge–Kutta scheme with an adaptive step size) are provided and the relative advantages of the MTL-based methods are highlighted with particular reference to the interpolating scheme used. Several useful future extensions of the method are also briefly touched upon.

2. The methodology

For most nonlinear problems in structural dynamics (governed by partial differential equations), one finally has to deal with a set of discretized, second-order, nonlinear ODEs following a projection technique, such as Rayleigh–Ritz, Galerkin or wavelet-Galerkin. This discretized system of nonlinear ODEs may be written as

$$\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{Q(\{X\}, \{\dot{X}\}, t)\} + \{F(t)\}. \quad (1)$$

In the above vector equations, $[C]$ and $[K]$ stand for the damping and stiffness matrices respectively, $Q(\{X\}, \{\dot{X}\}, t)$ is a nonlinear vector function of its first two arguments, $\{F(t)\}$ is an externally applied force vector and $\{X\}, \{\dot{X}\} \in \mathfrak{R}^n$ are respectively the n -dimensional displacement and velocity vectors. Let the initial condition vector to integrate Eq. (1) be denoted as $\{X(t_0)\} \triangleq \{X_0\}$ and $\{\dot{X}(t_0)\} \triangleq \{\dot{X}_0\}$. Now, consider a closed-open sub-interval $I_1 = [t_0, T | T > t_0)$ of the time axis and let it be ordered into p smaller intervals as $t_0 < t_1 < \dots < t_p = T$. Presently, the time step-size $h_i = t_{i+1} - t_i = h$ is taken to be constant only for purposes of a simple exposition of the basic concepts. The objective is now to derive an n -dimensional, linearized and easily integrable system of second-order ODEs, whose response should, in a sense, be ‘close’ to the response of Eq. (1) over I_1 . In particular, following the concept of transversal linearization ([11,12,15,26]), it is intended to derive the linearized ODEs such that the linearized and nonlinear vector fields remain identical at all the $(p + 1)$ points of discretization (grid points), viz. t_0, t_1, \dots, t_p .

Let the nonlinear and linearized flows, as parameterized by time t , be denoted by ϕ_t and $\bar{\phi}_t$ respectively. Thus, for the nonlinear flow, ϕ_t is essentially a C^k ($k \geq 0$) mapping (diffeomorphism) on the associated (compact) manifold M :

$$\phi_t(\{X, \dot{X}\}) : M \times R \rightarrow M \tag{2}$$

Moreover, the following relations hold:

$$\phi_{t_0} \triangleq \phi_0 = id_M, \phi_{h_p}^0 \phi_{h_{p-1}}^0 \phi_{h_{p-2}}^0 \dots \phi_{t_0}^0 = \phi_{t_p} \tag{3}$$

provided that the vector field is autonomous (note that a two n -dimension non-autonomous vector field may be posed as $(2n + 1)$ -dimensional autonomous one). The set of all such C^k diffeomorphisms, $\phi_t(X, \dot{X})$, under the operation of composition ‘ \circ ’ form a group, G_ϕ . Now, the second of Eq. (3) may be interpreted in terms of two different group actions. Whilst the first is the R -action defined as

$$R \rightarrow G_\phi : t \rightarrow \phi_t, \tag{4a}$$

the second is known as the Z -action

$$Z \rightarrow G_\phi : j \rightarrow \phi_{t_j}. \tag{4b}$$

Here, Z denotes the set of all integers. In what follows, an MTL method is outlined whose purpose is

- (i) derive a system of conditionally linear ODEs whose solutions have the same Z -action for any $t_j, j \in Z$.
- (ii) to require the solutions of the conditionally linear ODEs to have a ‘close’ R -action as the solutions of the nonlinear system.

At this stage, it is noted that the MTL-based linearized ODEs are only made to be conditionally linear, being conditioned upon the precise resemblance of the nonlinear and linearized vector fields at the grid points, t_j , and not necessary elsewhere. In other words, this implies that the conditionally linear flow would have to transversally intersect the nonlinear flow at least at the grid points. However, unlike the LTL method proposed earlier [15,26], an additional condition of the linearized solution remaining everywhere ‘close’ (with respect to a distance metric) to the nonlinear flow will be imposed during the development of the MTL strategy.

As with any other numerical technique, the entire interval of the time axis over which the ODEs are to be integrated, needs to be discretized. Presently the time interval of interest is ordered into overlapping N sub-intervals as

$$I = \{I_1, I_2, \dots, I_N\} = \{(t_0, t_1, \dots, t_p), (t_p, t_{p+1}, \dots, t_{2p}), \dots, (t_{(N-1)p}, t_{(N-1)p+1}, \dots, t_{Np})\}.$$

The integer N also denotes the number of times the MTL-based linearization procedure has to be applied so that an approximate solution over the desired interval I may be found and p denotes the number of grid points in each sub-interval (excluding the point at which the initial conditions are defined) covered by each application of the MTL procedure. To make the rest of the presentation simpler without losing generality, $p = 2$ is chosen for the discussion to follow.

In order to further motivate the development of the MTL method, it is instructive to expand the response vectors in a Taylor series. For instance, an expansion of the j th displacement component

${}^jX(t + \Delta t)$ in terms of the temporal increment Δt readily yields

$${}^jX(t + \Delta t) = {}^jX(t) + {}^j\dot{X}(t)\Delta t + {}^j\ddot{X}(t)\frac{\Delta t^2}{2} + \dots \tag{5}$$

In other words, the solution space of approximated displacement and velocity variables may be thought of as being spanned by the basis polynomials $1, \Delta t, \Delta t^2, \Delta t^3, \dots$ and so on. However, in the multi-step strategy to be developed, a direct use of Taylor’s expansion is not quite preferable, as it would necessitate repeated differentiations of the vector fields. Moreover, for $|\Delta t| \ll 1$, these basis polynomials are of different orders of magnitude; this may be responsible for introducing ill-conditioning (thereby leading to Gibbs phenomena) in the algorithm, especially as the number of terms in the expansion (i.e., the integer parameter p) increases. Nevertheless it is instructive to build the first version of the MTL method based on an expansion similar to that in Eq. (5). Thus an inspection of Eq. (5) reveals that the following is a hierarchical and complete expansion for the j th component, ${}^jQ(X, \dot{X}, t)$, of the nonlinear part of the vector field:

$${}^jQ_E(X, \dot{X}, t) = {}^jb_0 + {}^jb_1(t - t_0) + {}^jb_2(t - t_0)^2 + {}^jb_3(t - t_0)^3 + \dots \tag{6}$$

provided that the expansion is taken to infinity and that the vector field is interpreted as a functional of t . In order that the above expansion can reproduce the originally nonlinear vector function ${}^jQ(X, \dot{X}, t)$, the coefficient $b_i, i \in Z$ are so determined as to satisfy the interpolating property:

$${}^jQ_E(X_k, \dot{X}_k, t_k) = {}^jQ(X_k, \dot{X}_k, t_k) \triangleq {}^jQ_k, \tag{7}$$

where, $X_k \triangleq X(t_k), \dot{X}_k \triangleq \dot{X}(t_k)$. Thus, for developing the MTL method with $p = 2$, the first step is to write down the following approximation for ${}^jQ(X, \dot{X}, t)$, valid over the interval $(t_0, t_2]$:

$${}^jQ(X, \dot{X}, t) \approx {}^jQ_M^{(3)}(X_k, \dot{X}_k, t|k = 0, 1, 2) = {}^jb_0 + {}^jb_1(t - t_0) + {}^jb_2(t - t_0)^2. \tag{8}$$

Following condition (7) one readily has

$${}^jb_0 = {}^jQ_0. \tag{9a}$$

The other coefficient may be found from

$$\begin{bmatrix} h & h^2 \\ 2h & (2h)^2 \end{bmatrix} \begin{Bmatrix} {}^jb_1 \\ {}^jb_2 \end{Bmatrix} = \begin{Bmatrix} {}^jQ_1 - {}^jQ_0 \\ {}^jQ_2 - {}^jQ_0 \end{Bmatrix} \tag{9b}$$

while Eq. (9a) is an explicit relation for jb_0 , the other two coefficients jb_1 and jb_2 may be explicitly found from Eq. (9b) once the discretized displacement vector components $X(t_1) \triangleq X_1, X_2$ and the corresponding velocity components \dot{X}_1, \dot{X}_2 are known. In other words, one has

$${}^jb_1 = {}^jb_1(X_1, X_2, \dot{X}_1, \dot{X}_2), \quad {}^jb_2 = {}^jb_2(X_1, X_2, \dot{X}_1, \dot{X}_2). \tag{10}$$

The second level MTL-based (i.e., $p = 2$) linearized system of ODEs corresponding to the nonlinear Eqs. (1) then takes the form

$$\{\ddot{Y}\} + [C]\{\dot{Y}\} + [K]\{Y\} = \{Q_M^{(3)}(X_k, \dot{X}_k, t|k = 0, 1, 2)\} + \{F(t)\}. \tag{11}$$

It is noted that the first vector function on the right-hand side of Eq. (11) may be construed as a conditionally known, equivalent forcing function, which replaces the nonlinear part of the vector field.

Given that the approximation for ${}^jQ(X, \dot{X}, t)$, as in Eqs. (8) and (9), is essentially based on (polynomial) interpolation similar to a Taylor expansion, such an implementation of the MTL method will henceforth be referred to as the Taylor-MTL. However, there exists a possibility of ill-conditioning and consequent numerical corruption of the Taylor-MTL method with an increase in p , especially for $|t - t_0| \ll 1$. In such a case, ill-conditioning occurs as the basis polynomials, $1, (t - t_0), (t - t_0)^2, \dots$, are of different orders of magnitudes. In order to bypass the problem, one may make use of the Lagrangian interpolating polynomials. Thus restricting attention to the first sub-interval characterized by the grid points $I_1 = \{t_0, t_1, t_2\}$ with $p = 2$, the MTL-based approximation for ${}^jQ(X, \dot{X}, t)$ may be written as

$${}^jQ_M^{(3)}(X_l, \dot{X}_l, t|l = 0, 1, 2) = \sum_{k=0}^2 P_k(t) {}^jQ_k, \tag{12a}$$

where the Lagrangian polynomials (quadratic for $p = 2$) $P_k(t)$ are given by

$$P_k(t) = \prod_{\substack{l=0 \\ l \neq k}}^2 \frac{(t - t_l)}{(t_k - t_l)}. \tag{12b}$$

It is of interest to note that the discretized nonlinear vector $\{{}^jQ_k\}$ may be thought of as a set of linear functionals $\{L_k|k = 0, 1, 2\}$ acting on the elements (continuous functions) of the vector space V_3 spanned by the polynomial basis set $\{P_k(t)\}$ such that

$$L_k({}^jQ_M^{(3)}(X_l, \dot{X}_l, t|l = 0, 1, 2)) = ({}^j)Q_k. \tag{13}$$

These functionals form a basis for the 3D vector space V_3^* , which is dual to V_3 [24] and satisfy the well-known relations:

$$L_k(P_l(t)) = \delta_{kl}, \quad P_k(t_l) = \delta_{kl}, \quad k, l \in [0, 2], \tag{14}$$

where δ denotes the Kronecker delta. The above relations imply duality of the two basis sets. Adaptation of Eqs. (12)–(14) for any $p \geq 1$ is straightforward.

Even though, interpolating Taylor and Lagrangian polynomials (ITPs and ILPs) probably provide conceptually simplest routes to implement the MTL method, they are certainly not the only (or, even the most efficient) way of doing so. Indeed, a more general setting for the approximation of ${}^jQ(X, \dot{X}, t)$ is obtainable as

$${}^jQ(X, \dot{X}, t) \cong {}^jQ_M(t) = \sum_{k=0}^p {}^jQ(X_k, \dot{X}_k, t) \phi_k(t), \tag{15}$$

where the basis set $\{\phi_k|k = 0, \dots, p\}$, indexed on I_1 , must satisfy $\phi_k(t_l) = \delta_{kl}$, and $X_k = X(t_k), \dot{X}_k = \dot{X}(t_k)$. Preferably, one should also have the normalization condition $\int_0^{ph} \phi_k(t) dt = 1$. In addition to ILPs, distributed approximating functionals (DAFs) [16] and interpolating wavelets or interpolets [17] are a few possible choices for $\{\phi_k\}$. Choosing such basis functions may help accurately represent highly localized and infinitely smooth response characteristics in both

physical and Fourier spaces. Moreover, the parameters of DAF basis functions could be tuned to simulate any band-pass filter accurately. It has also been pointed out by Wei et al. [16] that DAF basis functions work very accurately within a global or mesh-free framework. Accordingly the precise choice of the basis functions should be decided by the nature of solutions that the user wishes to capture and the type of solution framework (global or local) being employed. In fact, it is this inherent non-uniqueness of the MTL method that should enable it to efficiently handle the dynamics of a wider variety of engineering applications. However, the attention of the present work will only be restricted to transversal linearization through ITPs and ILPs and a more comprehensive coverage of the MTL method (including these issues) along with relevant applications will be taken up elsewhere.

The solution of the linearized Eq. (11) may be performed in a $2n$ -dimensional (augmented) phase space by introducing the $2n$ -dimensional vector: $\hat{Y} = \{\{Y_d\}^T, \{Y_v\}^T\}^T \in \mathfrak{R}^{2n}$. Moreover, let $\hat{X} = \{\{X_d\}^T, \{X_v\}^T\}^T \in \mathfrak{R}^{2n}$, where the subscripts ‘ d ’ and ‘ v ’ are respectively used to denote the displacement and velocity vectors. Thus, in the discussion to follow, the vectors X_d and X_v would be used interchangeably with the vectors X and \dot{X} respectively. Eq. (11) may then be recast as

$$\{\dot{\hat{Y}}\} = [A]\{\hat{Y}\} + \{\hat{F}(\hat{X}_k, t)\}, \tag{16}$$

where $[A]$ is a $2n \times 2n$ -dimensional coefficient matrix given by

$$[A] = \begin{bmatrix} [0] & [I] \\ -[K] & -[C] \end{bmatrix} \tag{17}$$

and $\{\hat{F}\}$ is the $2n$ -dimensional conditional forcing vector, given by

$$\{\hat{F}\} = \left\{ \begin{array}{c} \{0\} \\ \{Q_M^{(3)}(\hat{X}_k, t)\} + \{F(t)\} \end{array} \right\}, \quad k = 1, 2. \tag{18}$$

In the above equations $\{0\}$ is an n -dimensional zero vector, $[0]$ is an $n \times n$ -dimensional zero matrix and $[I]$ is an $n \times n$ -dimensional identity matrix.

A conditional solution of Eq. (16) subject to the initial conditions $(Y_{d0}, Y_{v0}) = (X_{d0}, X_{v0}) = (X_0, \dot{X}_0) \in \mathfrak{R}^{2n}$ is written as

$$\hat{Y}(t) = [\phi(t, t_0)] \left[\{\hat{Y}_0\} + \int_{t_0}^t [\phi^{-1}(s, t_0)] \{\hat{F}(s)\} ds \right], \tag{19}$$

where $\hat{Y}_0 \triangleq \begin{Bmatrix} Y_{d0} \\ Y_{v0} \end{Bmatrix}$ and $\phi(t, t_0)$ is the $2n \times 2n$ -dimensional fundamental solution matrix (FSM), obtainable from the matrix exponentiation:

$$[\phi(t, t_0)] = \exp\{[A](t - t_0)\}. \tag{20}$$

For preservations of certain invariants of motion (such as the Hamiltonian functional for flows preserving the phase space volume), it is sometimes required that the matrix exponentiation be evaluated nearly exactly (i.e., to a vary high accuracy). In such a case, an eigensolution of the

matrix A is performed to arrive at its Jordan canonical form

$$J = P^{-1}AP = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & J_\mu \end{bmatrix}, \quad (21a)$$

where P is the (column-wise arranged) invertible matrix of eigenvectors and J_1, \dots, J_μ are the Jordan blocks associated with the different eigenvalues of A . Then the matrix exponentiation is computable via the identity

$$\exp(At) = P \exp(Jt)P^{-1}. \quad (21b)$$

It is recalled that exponentiation, $\exp(Jt)$, of the Jordan matrix J may be readily performed by writing J as a sum of diagonal and nilpotent matrices. It is also noted that the present development of the MTL method requires that the eigensolution of A is done only once during the entire process of numerical integration. Nevertheless, given that most of the problems of relevance in structural dynamics are non-Hamiltonian (i.e., with viscous damping), a more expedient way to compute matrix exponentiation may be to directly use a Taylor scheme:

$$[\phi(t, t_0)] = [\bar{I}] + [A](t - t_0) + [A]^2 \frac{(t - t_0)^2}{2!} + [A]^3 \frac{(t - t_0)^3}{6!} + \dots, \quad (21c)$$

where $[\bar{I}]$ is a $2n \times 2n$ -dimensional identity matrix. A detailed account of 19 different schemes of computing the matrix exponential as well as their limitations is provided in a recent article [18]. Now, within the MTL framework, the following constraint conditions are now imposed:

$$\begin{aligned} Y_k &= Y_{dk} \triangleq Y_d(t_k) = X_{dk}, \\ \dot{Y}_k &= Y_{vk} \triangleq Y_v(t_k) = X_{vk}, \quad k = 1, 2. \end{aligned} \quad (22)$$

Use of Eq. (19) in Eq. (22) results in $4n$ coupled nonlinear algebraic equations in as many unknown quantities, viz. $X_{d1}, X_{d2}, X_{v1}, X_{v2}$ for $p = 2$. One may thus arrive at desired solution vector by determining the zeros of Eq. (22), say with a Newton–Raphson or a nonlinear iterative technique. In general, for the p th level MTL method, one may have to solve for a maximum of $2pn$ nonlinear algebraic equations. Attention may now be focused on the tangent maps of nonlinear and MTL-based linearized vector fields, respectively denoted as $T_{\hat{X}_i} V_n$ and $T_{\hat{X}_i} V_l$, where $\hat{X}_i = \begin{Bmatrix} X_{di} \\ X_{vi} \end{Bmatrix}$ is a point in the $2n$ -dimensional phase space at which the tangent maps are constructed. Thus, one has (using Eq. (1))

$$\delta \hat{X}|_{\hat{X}=\hat{X}_i} = [A]\{\delta \hat{X}\} + \left[D_{\hat{X}=\hat{X}_i} \left\{ \begin{matrix} \{0\} \\ \{Q(\hat{X}, t)\} \end{matrix} \right\} \right] \{\delta \hat{X}\} \quad (23a)$$

and

$$\delta \hat{Y}|_{\hat{Y}=\hat{X}_i} = [A]\{\delta \hat{Y}\}, \quad (23b)$$

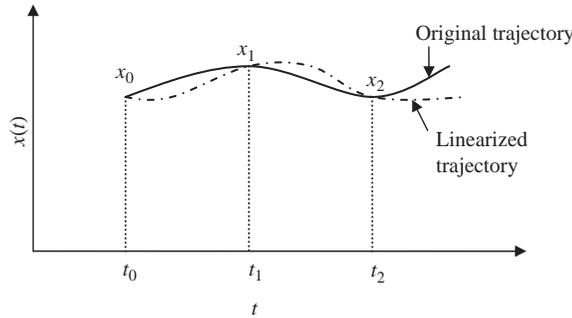


Fig. 1. A schematic representation of the MTL-based ($p = 2$) approximation for 1D dynamical system.

where, $\delta \hat{X}, \delta \hat{Y} \in \mathfrak{R}^{2n}$ are infinitesimally admissible variations on $\hat{X} \triangleq \begin{Bmatrix} X_d \\ X_v \end{Bmatrix}$ and $\hat{Y} \triangleq \begin{Bmatrix} Y_d \\ Y_v \end{Bmatrix}$ respectively and $D_{\hat{X}}$ stands for the Jacobian of the associated vector with respect to the components of \hat{X} . The tangent vectors, $T_{\hat{X}_i} V_n$ and $T_{\hat{Y}_i} V_l$, are respectively given by the vector fields of Eqs. (23a) and (23b). It is thus clear that for nearly all non-trivial cases (i.e., except when $\hat{X}(t) = \hat{Y}(t) = \{0\}$), one has $T_{\hat{X}_i} V_n \neq T_{\hat{X}_i} V_l$. In other words, $T_{\hat{X}_i} V_n$ and $T_{\hat{X}_i} V_l$ are transversal to each other for nearly any choice of $\hat{X}_i \in \mathfrak{R}^{2n}$. Zeros of the algebraic Eqs. (22) may then be interpreted as the points of transversal intersections of the nonlinear and linearized solution manifolds in the $2n$ -dimensional phase space. A schematic representation of the idea for a 1D dynamical system is shown in Fig. 1 for $p = 2$.

2.1. Error estimates

A straightforward estimation of local and global error orders in the MTL-based family of linearization procedures is possible based on appropriate Taylor expansions of the displacement and velocity components. To facilitate such an exercise, the governing nonlinear equations of motion are re-written in the form

$$\ddot{X} = L(X, \dot{X}, t) + Q(X, \dot{X}, t), \tag{24}$$

where L and Q are respectively the linear and nonlinear vector functions of $X, \dot{X} \in \mathfrak{R}^n$. In particular, a comparison with the governing equation yields the following form for the linear operator, L :

$$L(X, \dot{X}, t) = -[C]\{\dot{X}\} - [K]\{X\} + \{F(t)\}. \tag{25}$$

Following the p th level MTL concept, the vector function $Q(X, \dot{X}, t)$ may be related to its linearized counterpart, $Q_M^{(p)}(X_k, \dot{X}_k, t|k \in [0, p])$, through the equation

$$Q(X, \dot{X}, t) = Q_M^{(p)}(X_K, \dot{X}_K, t) + R_M^{(p)}(t), \quad X_K \triangleq X(t_K), \quad \dot{X}_K \triangleq \dot{X}(t_K). \tag{26}$$

The last term, $R_M^{(p)}(t)$, on the right-hand side of the above equation is the conditionally determined remainder term of order $(t - t_0)^{p+1}$ for the p th level MTL procedure. At this stage, the j th component, ${}^j X(t) \in \mathfrak{R}$, of the displacement vector $X(t)$ may be Taylor expanded as

$${}^j X(t + \Delta t) = {}^j X(t) + {}^j \dot{X}(t)\Delta t + {}^j \ddot{X} \frac{\Delta t^2}{2!} + {}^j \ddot{\ddot{X}} \frac{\Delta t^3}{3!} + \dots + \frac{d^r({}^j X(t))}{dt^r} \cdot \frac{(\Delta t)^r}{r!} + R_r(t), \tag{27}$$

where $R_r(t)$ is the remainder term of order $(\Delta t)^{r+1}$. Appropriate continuity, differentiability and bounded-ness criteria are implicitly imposed on the dependent variables and functions in the above equation and the ones to follow. Now, it is noted that ${}^j\ddot{X}(t)$ in the third term of the right-hand side of Eq. (27) may be represented as

$${}^j\ddot{X}(t) = {}^jL(X, \dot{X}, t) + {}^jQ_M^{(p)}(X_k, \dot{X}_k, t) + {}^jR_M^{(p)}(t). \quad (28)$$

Similarly, one may write

$${}^j\ddot{X}(t) = {}^jL_X(X, \dot{X}, t)\dot{X} + {}^jL_{\dot{X}}(X, \dot{X}, t)\ddot{X} + {}^jL_t(X, \dot{X}, t) + {}^j\dot{Q}_M^{(p)}(X_k, \dot{X}_k, t) + {}^j\dot{R}_M^{(p)}(t), \quad (29)$$

where

$${}^jL_X(\cdot) = \frac{\partial({}^jL(\cdot))}{\partial X}, \quad {}^jL_{\dot{X}}(\cdot) = \frac{\partial({}^jL(\cdot))}{\partial \dot{X}}, \quad {}^jL_t(\cdot) = \frac{\partial({}^jL(\cdot))}{\partial t}$$

and

$${}^j\dot{Q}_M^{(p)}(\cdot) = \frac{d({}^jQ_M^{(p)})}{dt}.$$

It may be observed that Eqs. (28) and (29), conditionally determining ${}^j\ddot{X}(t)$ and ${}^j\dot{X}(t)$ respectively, are exact subject to knowledge of the discretized state vectors $\{X_k\}$ and $\{\dot{X}_k\}$ at $t = t_k, k \in [1, p]$. It may be recalled that these vectors are determined via constraint (algebraic) Eqs. (22) within the MTL framework. Similar expressions for higher-order derivatives $d^n/dt^n({}^jX(t))$ may also be written. The MTL-based approximations for these derivatives (second-order and higher) may be recovered by removing the last remainder term, ${}^jR_M^{(p)}(t)$, or its derivatives from Eqs. (28), (29), etc.

In order to determine the local and global error orders in displacement computations through the p th level MTL, it suffices to substitute expressions (28), (29) and those for higher-order derivatives into the Taylor expression (27) and determine the lowest order (in terms of the integral exponents of Δt) of the terms containing $R_M^{(p)}(t)$ or its derivatives. For further elaboration, it is convenient to consider, say, $p = 2$, which corresponds to $R_M^{(2)}(t) \equiv O(\Delta t^3)$. Let $r \geq p + 1$ in Eq. (27). Substitution of Eq. (28) for ${}^j\ddot{X}(t)$ in Eq. (27) leads to a term of the form ${}^jR_M^{(p)}(t)\Delta t^2/2! \equiv O(\Delta t^5)$, which may be verified as one of the lowest order terms involving $R_M^{(p)}(t)$. Similarly, substitution of Eq. (29) for ${}^j\dot{X}(t)$ in Eq. (27) leads to a term ${}^j\dot{R}_M^{(p)}(t)\Delta t^3/3! \equiv O(\Delta t^5)$ (noting that ${}^j\dot{R}_M^{(p)}(t) \equiv O(\Delta t^2)$), which is another lowest order term involving ${}^jR_M^{(p)}(t)$ or its derivatives. Continuing this way, one may conclude that the local displacement error, $R_d^{(2)}$, via the second level MTL method is given by

$$R_d^{(2)} \equiv O(\Delta t^5). \quad (30)$$

Generalizing in this way, one may readily obtain the local displacement error order for p th ($p \geq 1$) level MTL method as

$$R_d^{(p)} \equiv O(\Delta t^{p+3}). \quad (31)$$

A similar argument for the velocity error (local) yields

$$R_V^{(p)} \equiv O(\Delta t^{p+2}). \quad (32)$$

The global error orders would be one integral order less than their local counterparts.

It is noted that many dynamical problems in structural dynamics may not have a sufficiently differentiable response. In such cases, the error order arguments based on Taylor expansions would not remain valid. Even in such cases, the MTL-based linearization would satisfy the nonlinear vector fields at the points of discretization and thus would remain numerically implementable. This is however not true for many of the available integration procedures which need a Taylor-like expansion for construction of the discrete point-to-point mapping.

3. Numerical results

The usefulness and some of the relative advantages of a new integration scheme are best appreciated through its numerical applications for certain problem of relevance in structural dynamics. Since the proposed family of MTL schemes has a fairly straightforward implementation for oscillators of any dimension, only a few single degree of freedom (sdof) nonlinear oscillators are presently considered to keep the demonstration simple and accordingly bring out a few essential features of the proposed schemes. Thus, to begin with, consider a viscously damped, hardening Duffing (HD) oscillator of the form

$$\ddot{x} + 2\pi\epsilon_1\dot{x} + 4\pi^2\epsilon_2(x + x^3) = 4\pi^2\epsilon_3 \cos 2\pi t. \tag{33}$$

Here the nonlinear part of the vector field is given by cubic, scalar (symmetrical) function

$$Q(x, \dot{x}, t) = Q(x) = -4\pi^2\epsilon_2x^3. \tag{34}$$

The details of implementation of the MTL-Taylor scheme are provided below for purposes of a ready reference. The essence of applying the MTL-Lagrangian scheme remains precisely the same except for a modification of Eq. (36), wherein Lagrangian interpolating polynomials are used to approximately expand $Q(X, \dot{X}, t)$. The p th level linearized form based on MTL-Taylor of Eq. (33) may be written as

$$\ddot{y} + 2\pi\epsilon_1\dot{y} + 4\pi^2\epsilon_2y = 4\pi^2\epsilon_3 \cos 2\pi t - 4\pi^2\epsilon_2Q_M^{(p)}, \tag{35}$$

where, one has

$$Q_M^{(p)}(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)^2 + \dots + b_p(t - t_0)^p, \tag{36}$$

where, $b_0 = x^3(t_0); b_1; \dots, b_p$ are conditionally determined as linear functions of the cubes of discretized variables $x(t_k), k \in [1, p]$, by solving the system of p linear equations

$$\begin{bmatrix} (t - t_0) & (t - t_0)^2 & \dots & (t - t_0)^p \\ 2(t - t_0) & 2^2(t - t_0)^2 & \dots & 2^p(t - t_0)^p \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p(t - t_0) & p^2(t - t_0)^2 & \dots & p^p(t - t_0)^p \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ b_p \end{Bmatrix} = \begin{Bmatrix} x^3(t_1) - x^3(t_0) \\ x^3(t_2) - x^3(t_0) \\ \vdots \\ \vdots \\ \vdots \\ x^3(t_p) - x^3(t_0) \end{Bmatrix} \tag{37}$$

for any given time t . Solutions of the linearized system (Eq. (35)) follow the general scheme as outlined in the last section. Accordingly, Eq. (35) is first written in the state-space form

$$\begin{Bmatrix} \dot{y}_d \\ \dot{y}_v \end{Bmatrix} = \{\dot{Y}\} = [A]\{Y\} + \{\hat{F}(t, x_k^3) | k \in [1, p]\}. \tag{38}$$

The coefficient matrix $[A]$ is given by

$$[A] = \begin{bmatrix} 0 & 1 \\ -4\pi^2\varepsilon_2 & -2\pi\varepsilon_1 \end{bmatrix} \tag{39}$$

and the augmented, conditional force vector $\hat{F} \in \mathfrak{R}^2$ is given by

$$\{\hat{F}(t, x_k^3)\}^T = \{0, 4\pi^2\varepsilon_3 \cos 2\pi t - 4\pi^2\varepsilon_2 Q_M^{(p)}(x_k^3, t)\}. \tag{40}$$

In the above equations, one has $x_k \triangleq x(t_k), k \in [0, p]$. Denoting the fundamental solutions matrix (FSM) as

$$[\phi(t, t_0)] = \exp[A(t - t_0)]. \tag{41}$$

The solution to the ODEs (35) is then written as

$$\begin{Bmatrix} y_d(t) \\ y_v(t) \end{Bmatrix} = [\phi(t, t_0)] \left[\begin{Bmatrix} x_0 \\ \dot{x}_0 \end{Bmatrix} + \int_{t_0}^t [\phi^{-1}(s, t_0)] \{\hat{F}(s, x_k^3)\} ds \right]. \tag{42}$$

The desired nonlinear algebraic equations (constraint equations) to determine $x_m, m \in [1, p]$ are then written as

$$\begin{aligned} x_m = & \phi_{11}(t_k, t_0)x_0 + \phi_{12}(t_k, t_0)\dot{x}_0 + \phi_{11}(t_k, t_0) \left[\int_{t_0}^{t_k} \phi_{12}^{-1}(s, t_0)\hat{F}_2(s, x_k^3) ds \right] \\ & + \phi_{12}(t_k, t_0) \left[\int_{t_0}^{t_k} \phi_{22}^{-1}(s, t_0)\hat{F}_2(s, x_k^3) ds \right], \quad k = 0, \dots, p, \end{aligned} \tag{43}$$

where, \hat{F}_2 is the second component of the vector \hat{F} given by Eq. (40). Presently, a Newton–Raphson scheme is used to determine the roots of the system of Eqs. (43). Following the determination of x_m , the discretized velocity components, $\dot{x}_m \triangleq \dot{x}(t_m)$, may be recovered from the second component equation of the vector Eqs. (42) by noting that $y_{vk} \triangleq \dot{y}(t_k) = \dot{x}_k$ for all $k \in [0, p]$. The above procedure, applicable to the sub-interval I_1 may be similarly repeated over the following sub-intervals I_2, I_3, \dots and so on, to obtain the MTL-based approximation to the solution over any interval of the time axis.

In all the examples to follow, a uniform time step size $h = 0.01$ has been adopted unless explicitly stated otherwise. In Figs. 2 and 3, a couple of typically one-periodic orbits of the HD oscillator are plotted via MTL-Taylor and MTL-Lagrangian methods corresponding to $p = 2$ and 3 and compared with a sixth order explicit Runge–Kutta method [19], which is often considered to be the most efficient in the family of Runge–Kutta methods [20]. Even a visual comparison of Figs. 3(a) and (c) is enough to show that the MTL-Taylor scheme shows poorer correspondence with the Runge–Kutta scheme (which is the most accurate scheme in this example) than the

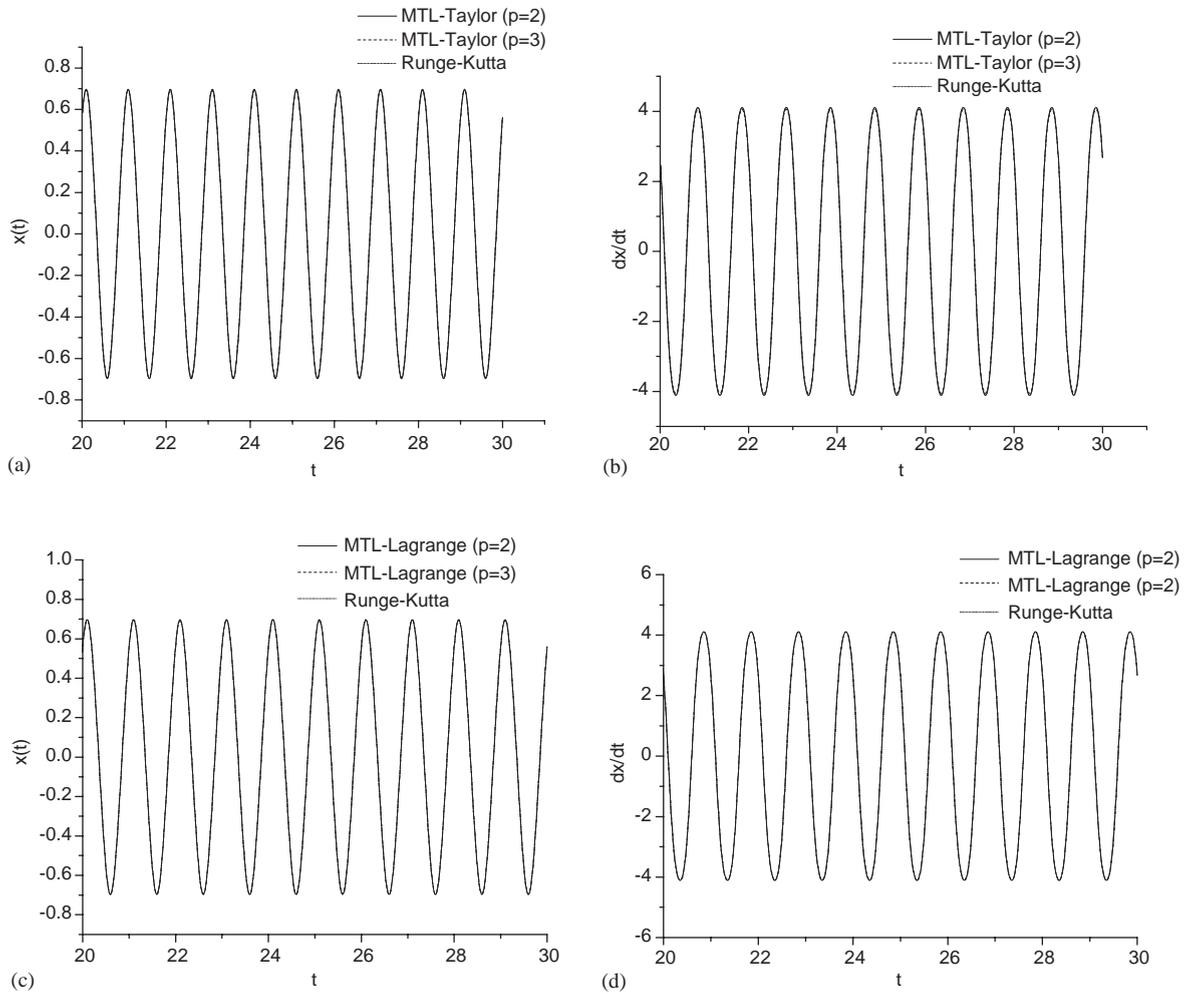


Fig. 2. A typically one-periodic regime of the hardening Duffing oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.3$: (a) displacement history via MTL-Taylor; (b) velocity history via MTL-Taylor; (c) displacement history via MTL-Lagrangian; (d) velocity history via MTL-Lagrangian.

MTL-Lagrangian scheme. The relative inaccuracy of the MTL-Taylor scheme with reference to its Lagrangian counterpart is also brought out in the phase plane diagrams of Figs. 3(e) and (f).

The well-known fact that chaotic trajectories are exponentially sensitive (in a local sense) to small numerical errors may be presently exploited in both to ascertain the relative numerical stability of the two versions of the MTL method. For instance, if one is integrating a dynamical system in its chaotic regime with two different numerical schemes (of comparable orders of accuracy), such that the integrated trajectories start from the same initial condition in both the cases, then the time interval over which the two trajectories remain ‘close’ to each other (with respect to a distance measure such as the Euclidean norm) is a measure of numerical stability (in terms of growth of local errors) of the methods. Figs. 4(e)–(f) show time histories of one such

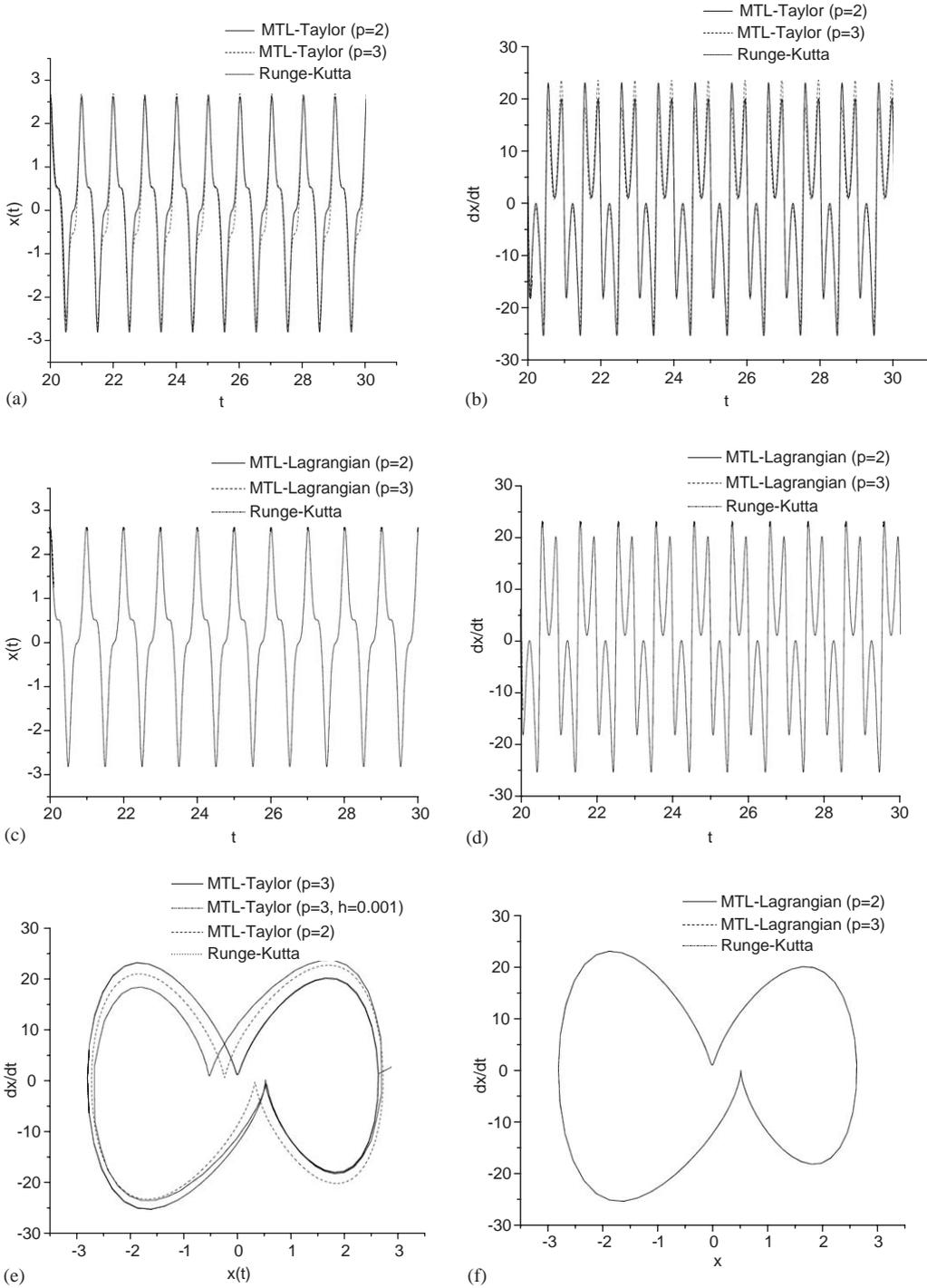


Fig. 3. A typically large one-periodic regime of the hardening Duffing oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 8.0$.

trajectory, integrated via MTL-Taylor ($p = 2$ and 3), MTL-Lagrangian ($p = 2, 3$ and 6) and sixth-order Runge–Kutta (all starting from the same initial condition). Even from a visual inspection of the time histories, it is clear that trajectories integrated via the MTL-Lagrangian scheme with $p = 2, 3$ remain consistently closer to those via Runge–Kutta than the ones integrated via MTL-Taylor with $p = 2, 3$ do. Figs. 4(e) and (f) show displacement and velocity histories of the same trajectory via sixth-order Runge–Kutta and MTL-Lagrangian with $p = 6$ (i.e., two methods of the same formal order of accuracy). The integrated chaotic trajectories via these two schemes show a remarkably closer correspondence (over longer durations) than the ones in Figs. 4(c) and (d). This lends numerical credence to an improved accuracy of the MTL-Lagrangian scheme with increasing values of p . Such an improvement is not observable with MTL-Taylor schemes since Taylor basis polynomials lead to ill-conditioning for higher values of p . That the closeness of chaotic orbits integrated through Runge–Kutta and MTL-Lagrangian is always far higher than the closeness of those integrated through Runge–Kutta and MTL-Taylor is more conspicuously seen in the phase plots of Figs. 5(a)–(f).

In order to have a better quantitative appreciation of the relative improvement of numerical accuracy of the MTL-Lagrangian family of schemes with increasing p , consider the instantaneous (Euclidean) error norm given by

$$E^{I,II}(t) = \frac{1}{2}((x^I(t) - x^{II}(t))^2 + (\dot{x}^I(t) - \dot{x}^{II}(t))^2)^{1/2}, \quad (44)$$

where the superscripts I and II respectively denote method I and method II, respectively. Owing to a locally exponential separation of nearby trajectories in the chaotic regime, it is anticipated that the above norm will grow with time for a chaotic orbit integrated from the same initial condition via two different numerical schemes. This is due to the difference in floating point operations employed by two such methods thereby leading to exponential magnifications of small numerical differences (owing to the positivity of the largest Lyapunov's exponent). This is unlike the case of periodic orbits wherein the error norm is expected to remain high in the initial stages and stabilize to a periodic solution (probably with smaller amplitudes) as the steady state is reached. Figs. 6(a) and (b) show the histories of relative error norms between MTL-Lagrangian schemes with different p while integrating a typically chaotic orbit of the hardening Duffing oscillator with the same initial condition. It is clear that relative errors between MTL-Lagrangian methods with higher values of p (say, with $p = 6$ and 7) grow much less than those between methods with lower values of p .

Being an implicit method which has the added advantage of ensuring the identity of the linearized and nonlinear vector fields at all the points of discretization, the MTL method remains accurate even under relatively larger time step sizes for which the approximations via explicit Runge–Kutta formulations may simply blow away. This observation is quite generic about MTL methods and is valid irrespective of the type of basis functions used to expand the nonlinear part of the vector field. Numerical demonstrations of this observation are provided using the MTL-Taylor method in Figs. 7 and 8 for a periodic case and a chaotic case, respectively. In other words, the MTL family of methods has a conspicuously higher numerical stability under higher step-sizes than the Runge–Kutta algorithms.

Yet another advantage of the MTL-based linearization is that a Hamiltonian system (nonlinear) remains so even after the linearization. However, the Hamiltonian for the linearized flow differs from that of the nonlinear flow in terms of certain integral powers of the time step, h ,

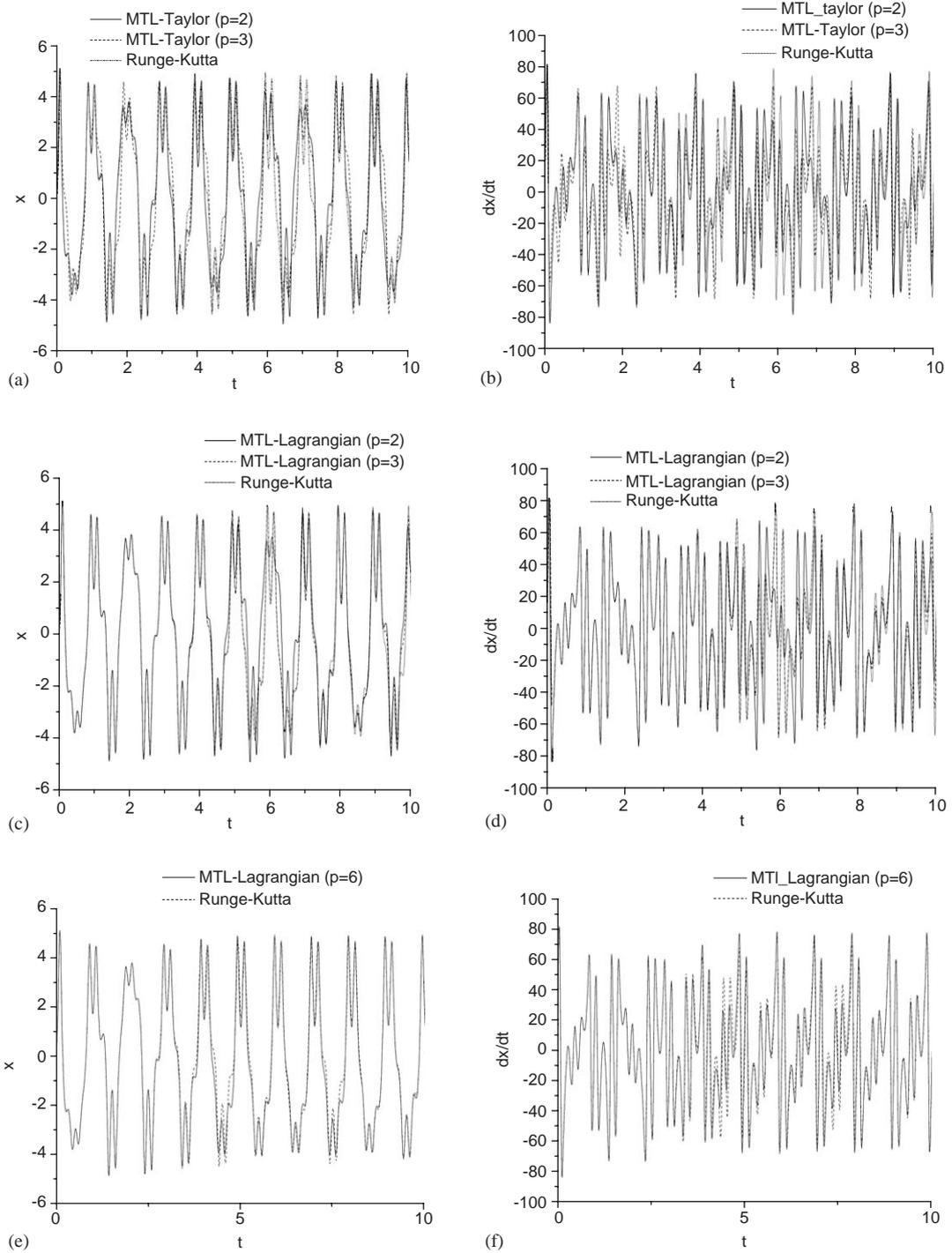


Fig. 4. Displacement and velocity histories of a typically chaotic regime of the hardening Duffing oscillator: $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 42.0$.

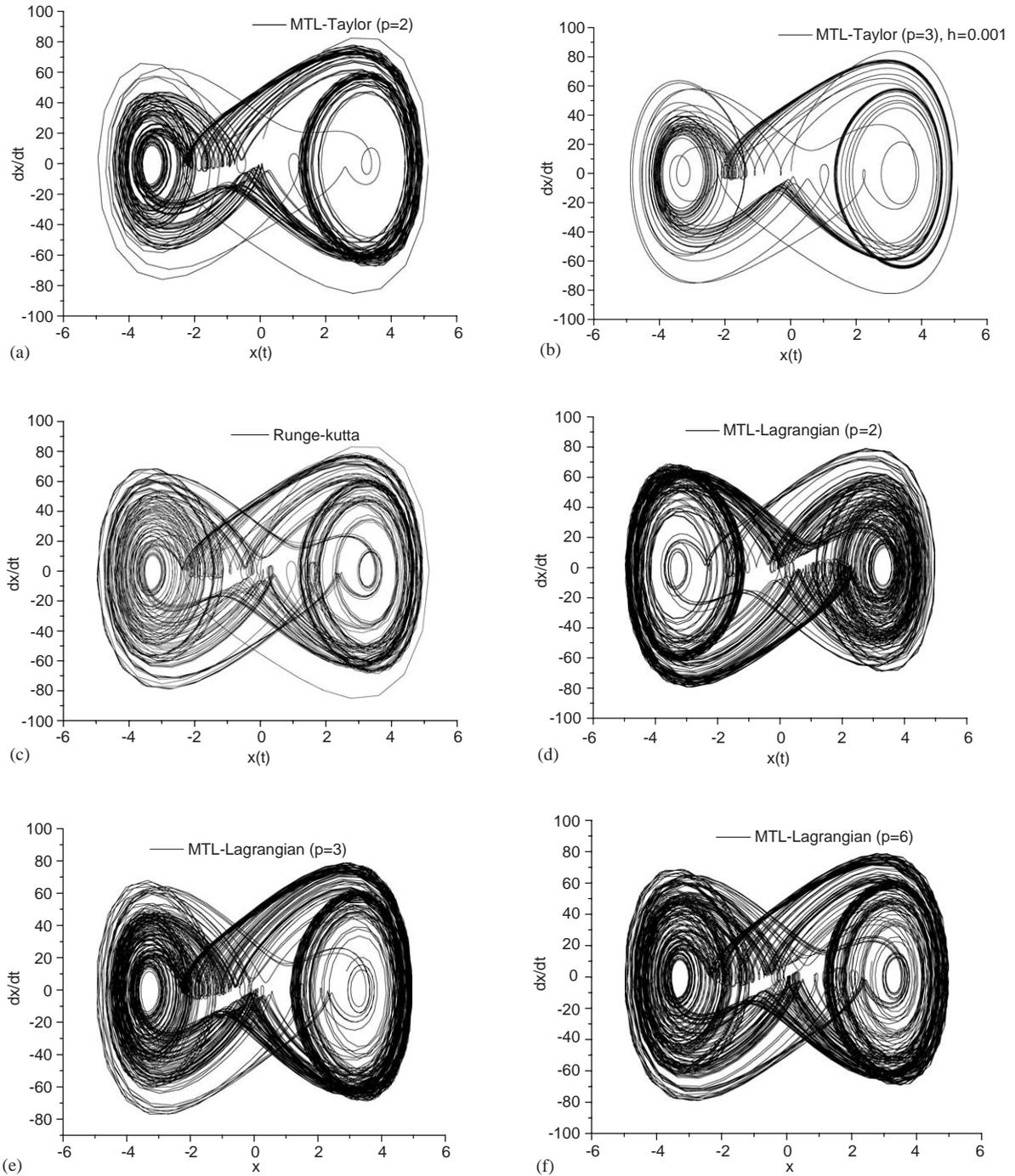


Fig. 5. Phase plots of a typically chaotic regime of the hardening Duffing oscillator: $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 42.0$.

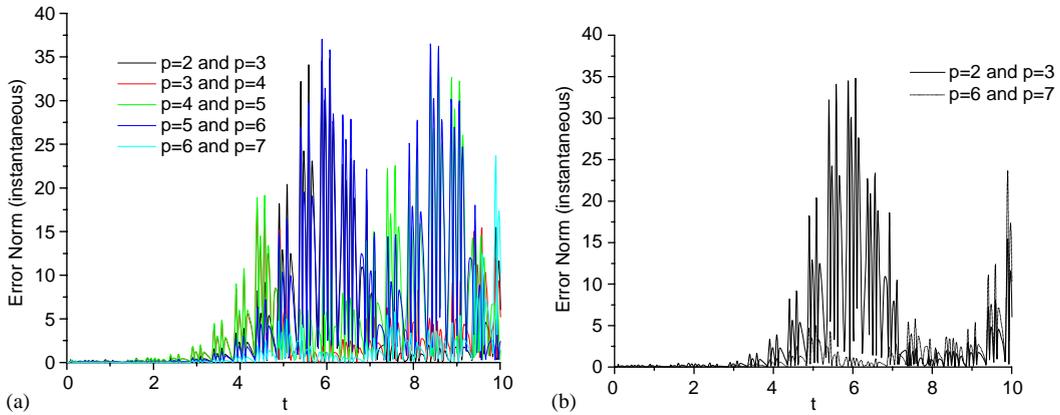


Fig. 6. Relative error histories between MTL-Lagrangian schemes with different p for a typically chaotic trajectory starting from the same initial condition, $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 42.0$, $x_0 = 0.1$, $\dot{x}_0 = 0.1$.

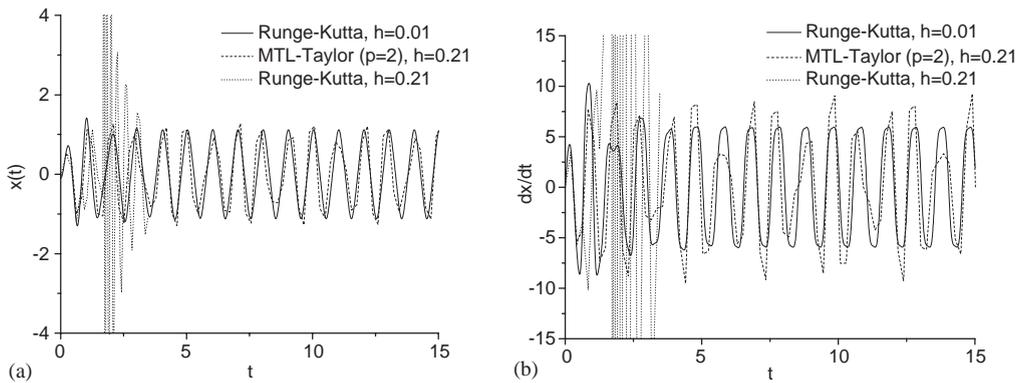


Fig. 7. Relative numerical stability of the MTL approach against different time step-sizes vis-à-vis the Runge-Kutta method for periodic orbit of the hardening Duffing oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 1.0$. Runge-Kutta method overflows for high time step-sizes.

and this difference is readily computable. Consider, for instance, undamped Duffing (hardening) oscillators, i.e., Eq. (33) with $\varepsilon_1 = 0$. An absence of viscous damping implies that the divergence of the vector field becomes zero and thus the highest Lyapunov exponent is higher. Thus if the Hamiltonian Duffing oscillator exhibits chaos, exponential separation of nearby trajectories is more pronounced compared to the same oscillator with positive viscous damping. As numerically illustrated earlier for chaotic solutions of a damped Duffing oscillator, the exponential divergence takes more time as the initial conditions are closer. Figs. 9(a) and (b) show the time histories of displacement of two chaotic trajectories of the Hamiltonian system, originated through the sixth level MTL-Lagrangian scheme ($p = 6$) and sixth-order Runge-Kutta method. It is evident that the exponential divergence of trajectories integrated through MTL takes longer to occur. It is conjectured that the preservation of the given form of the vector field at all the points of discretization (in the MTL method) is one of the main factors responsible for a slower divergence

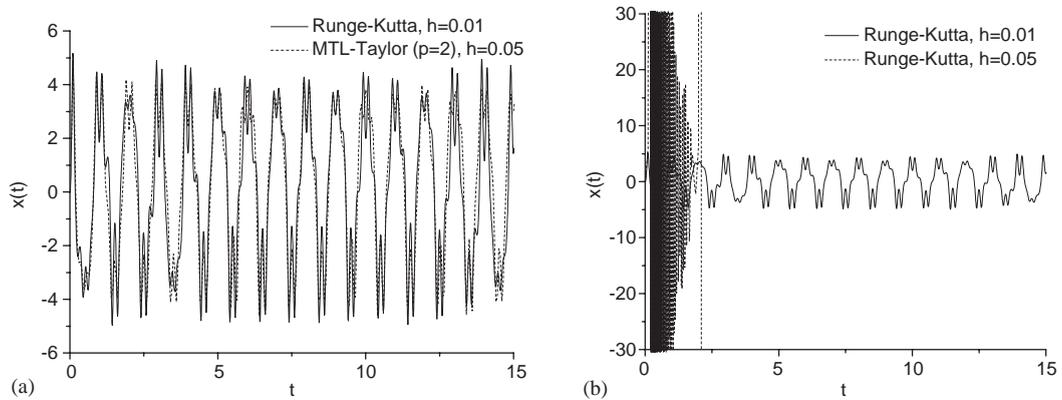


Fig. 8. Loss of accuracy of the Runge-Kutta method for high values of h for a chaotic regime of the hardening Duffing oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 42.0$. In this case, solutions via the Runge-Kutta method does not overflow, but converge to wrong values.

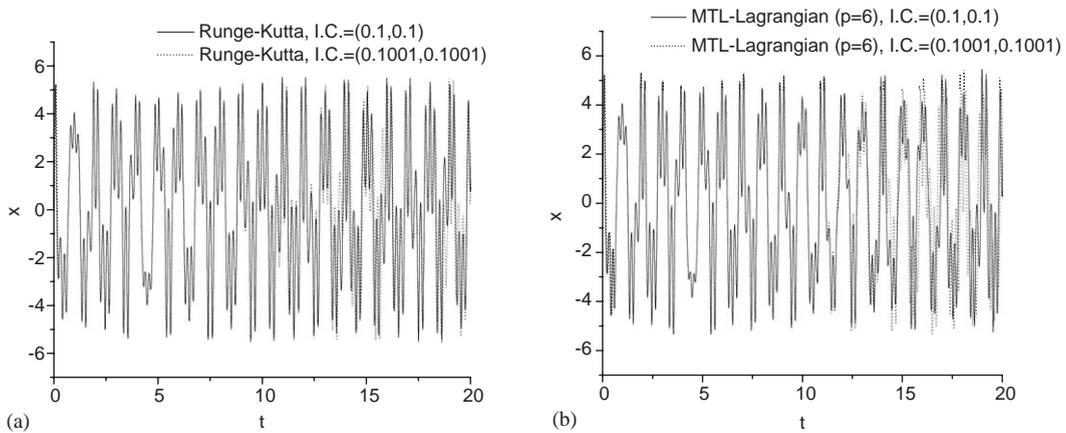


Fig. 9. A comparison of numerical robustness of the MTL ($p = 2$) vis-à-vis the Runge-Kutta method for two chaotic trajectories of the hardening, undamped Duffing oscillator starting with closely separated initial conditions $\varepsilon_1 = 0.0$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 42.0$; I.C. denotes the initial condition vector (x_0, \dot{x}_0) .

of nearby trajectories as compared to the Runge-Kutta method. As was noted previously, such a numerical stability of the MTL method is observable, even though in a smaller measure, when $\varepsilon_1 > 0$ as well (i.e., when dissipative chaos occurs).

As a second numerical example, a Duffing-Holmes' oscillator, which has two potential wells in the unperturbed (un-damped, unforced) phase plane, is considered. The oscillator is governed by the following second-order nonlinear ODE

$$\ddot{x} + 2\pi\varepsilon_1\dot{x} + 4\pi^2\varepsilon_2(x^3 - x) = 4\pi^2\varepsilon_3 \cos 2\pi t. \quad (45)$$

Derivations of the MTL-based linearized forms (via Taylor and Lagrangian schemes) and their subsequent solutions follow precisely the same steps as for the previous example and need not be

elaborated further. For sufficiently small values of the forcing amplitude parameter, ε_3 , the harmonically forced, damped oscillator exhibits a couple of egg-shaped 1-periodic orbits, symmetrically placed around the x -axis in the associated phase plane, $x - \dot{x}$. While Figs. 10(a) and (b) show these orbits via MTL-Taylor schemes with $p = 2$ and 3 as well as comparisons with the sixth-order Runge–Kutta method, Fig. 10(c) shows phase plots of the same orbits through MTL-Lagrangian schemes corresponding to $p = 3$ and 4. The relatively higher accuracy of the MTL-Lagrangian family is again clear even from visual comparisons of these figures. For higher values of ε_3 , chaotic solutions are possible due to homoclinic bifurcations of the perturbed separatrix and consequent diffusion of trajectories amongst the two potential wells. The strange attractor of a typically chaotic orbit, obtained via MTL-Taylor and Runge–Kutta methods, are plotted in Figs. 11(a) and (b). In this case too, the MTL-Lagrangian family of schemes may be shown to perform better. This is illustrated through the instantaneous error history plots reported in Figs. 11(c) and (d). Of particular interest is the drastic reduction of the relative error norm between MTL-Lagrangian schemes with $p = 6$ and 7 with reference to that between MTL-Lagrangian with $p = 2$ and Runge–Kutta.

Indeed, it has been numerically observed for the case of the hardening Duffing oscillator that the MTL-Taylor methods corresponding to fourth and higher levels require very small step sizes of the order of $0(10^{-4}, 10^{-5})$ for convergence. In fact, for highly oscillatory cases (such as in the chaotic regime), the Newton–Raphson method may not converge at all for MTL-Taylor schemes with $p \geq 4$. Thus, out of these two families of schemes, the MTL-Lagrangian family is preferable from the points of view of higher accuracy and numerical stability.

4. Discussion and concluding remarks

A family of implicit, multi-step transversal linearization (MTL) methods is proposed for numerically accurate, computationally efficient and semi-analytical integrations of nonlinear mechanical oscillators of significance in engineering dynamics. The basis of the linearization is the replacement of the nonlinear part of the vector field via a conditionally linearized one. Using a set of interpolating basis functions, the conditionally linearized replacement is derived such that it precisely reproduces the nonlinear part of the vector field at all the grid points. The conditional nature of this replacement arises from the fact that the coefficients of the interpolating basis functions are expressible in terms of the unknown, discretized state vectors at the grid points. A specific advantage of MTL methods lies in the flexibility with which the functional or operator interpolation scheme for the nonlinear part of the vector field can be chosen. Presently Taylor and Lagrangian basis polynomials are used and numerically explored for interpolating the nonlinear part of the vector field. Since tangent spaces of the nonlinear and conditionally linear vector fields are not identical (i.e., transversal) almost everywhere in the associated phase space, and, in particular, at the grid points, the unknown state vectors are determined by constructing a set of algebraic constraint equations, which ensure transversal intersections of the linearized and nonlinear solution manifolds. The constraint equations are in the form of a set of nonlinear (transcendental) algebraic equations, whose roots may be found through a Newton–Raphson or a nonlinear iterative method.

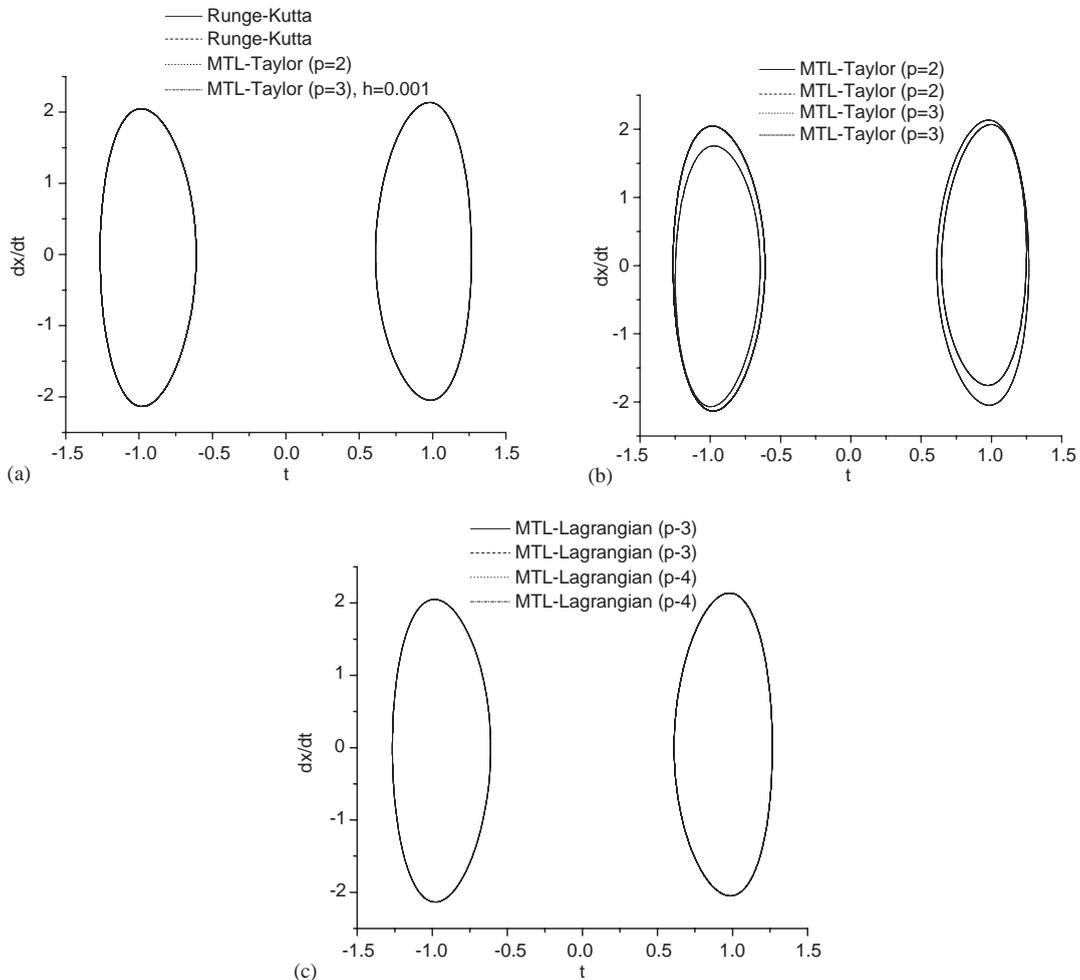


Fig. 10. Egg-shaped one-periodic orbits of the Duffing-Holmes' oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 0.5$, $\varepsilon_3 = 0.1$ ($h = 0.01$ unless otherwise mentioned). A loss of accuracy of MTL-Taylor ($p = 3$) vis-à-vis the MTL-Lagrangian for $h = 0.01$ is noted.

Simple estimates of error in displacement and velocity components, as computed via the MTL family methods, have been provided. While derivation of higher-order MTL methods is quite straightforward and merely involves increasing the order of interpolation, numerical instability may creep in for MTL-Taylor schemes beyond a certain order. It is however demonstrated that higher-order versions of the MTL-Lagrange method do not suffer from (or, is far less prone to) such drawbacks. The MTL family of methods has been shown to have a generically superior numerical stability than Runge–Kutta methods, especially for larger time step sizes. Yet another additional advantage of these methods is that the linearization is performed in such a way so as to preserve the (possibly) Hamiltonian nature of the oscillator. This aspect assumes particular importance for integrations performed over long durations. A host of numerical illustrations of the procedures for a couple of sdof oscillators, viz. the Duffing and the Duffing–Holmes'

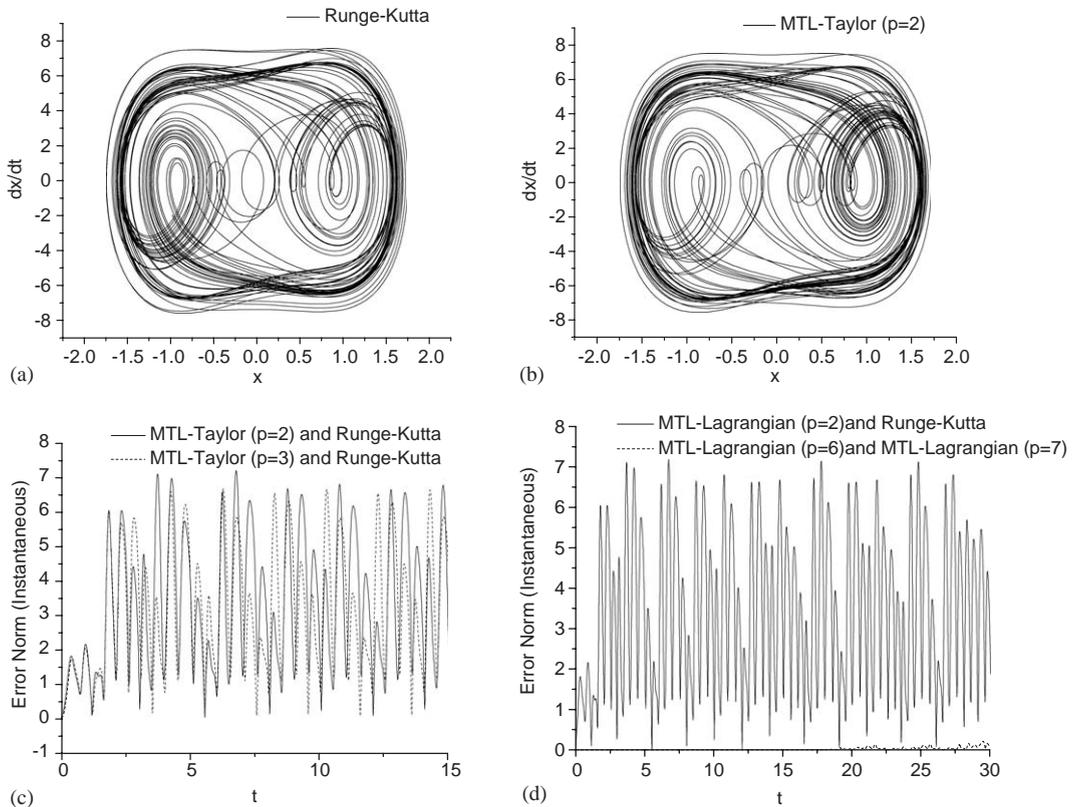


Fig. 11. A typically chaotic diffusion of the Duffing-Holmes' oscillator as predicted by MTL-Taylor ($p = 2$), Runge-Kutta and MTL-Lagrangian ($p = 2, 6$ and 7) methods; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 0.5$, $\varepsilon_3 = 0.5$.

oscillators, are provided to bring out some of their useful features. The numerical implementation of MTL methods remains precisely the same irrespective of the dimensionality of the oscillator concerned.

It is possible to exploit the non-uniqueness aspect of an MTL-based formulation and thus develop a MTL family of schemes using the Magnus formula within a Lie algebraic framework. In such a case, the nonlinear part of the vector field may be decomposed as products of time-dependent matrices times the response (displacement and velocity) vectors. The linearized dynamical system would then have (conditionally) parametric terms and its solution may be found, to any order of accuracy, using the Magnus or Fer formulas [21]. This extension would hopefully allow the MTL family further flexibility to adapt even better to complex dynamical situations. However, one difficulty with such an approach could be in the form of the significantly more computational overhead associated with repeated evaluations of the linearized stiffness matrix over each time step. A systematic study on such a form of the MTL method is now under progress. Another objective of immediate interest to the authors is to modify and apply a variation of the present scheme for numerical integrations of the dynamical equations of motion for beams undergoing finite strains [22]. In particular, given that such equations are formulated in terms of both displacement and rotational variables and that rotations (unlike displacements)

evolve on nonlinear manifolds, a direct adaptation of the MTL strategy proposed here for such problems is not possible. This is due to the fact that rotations cannot be interpolated in the same way as displacements [23]. However, nonlinear manifolds have locally vector space structures and this observation may be exploited to modify and extend the present formulations to such problems of considerable contemporary interest.

Acknowledgements

The first author wish to take this opportunity to express his gratitude to one of the referees for all the painstaking comments and suggestions. They have helped in (hopefully) improving the original form of the MTL method to a substantive extent.

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